Uncertain Quantities

7

7.1 DISCRETE UNCERTAIN QUANTITIES

Discrete Uncertain Quantity (UQ): a few, distinct values

Assign probability mass to each value (probability mass function).

A corporate planner thinks that uncertain market revenue, X, can be approximated by three possible values and their associated probabilities: P(X=10000) = 0.25, P(X=12000) = 0.50, and P(X=15000) = 0.25. For each outcome, the value, discrete probability mass, and cumulative probability are shown in Figure 7.1.

Figure 7.1 Market Revenue Discrete UQ

x	P(X=x)	P(X<=x)
$10,000	0.25	0.25
$12,000	0.50	0.75
$15,000	0.25	1.00

The probability mass function is shown in the following figure. An alternative way to display the data uses a cumulative distribution. As you move your eye from left to right on the cumulative value scale, the probability mass is added at the appropriate values. A discrete UQ has a cumulative distribution that looks like stair steps, and each step corresponds to the probability mass.

Figure 7.2 RandDiscrete Example Probability Mass Function

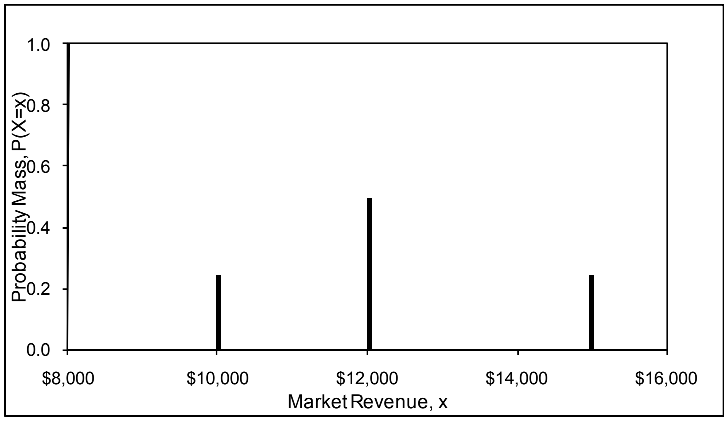
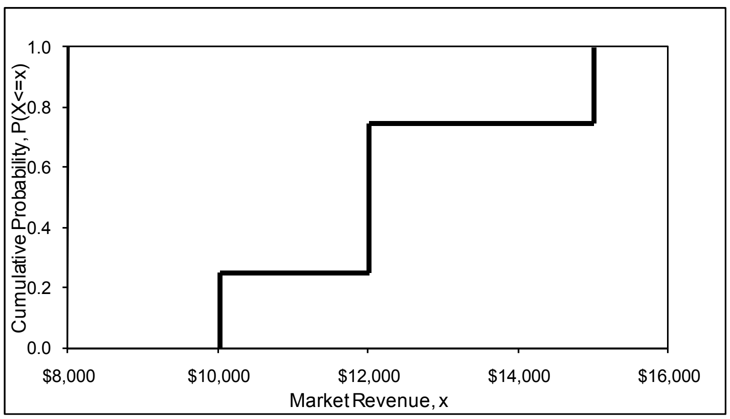


Figure 7.3 RandDiscrete Example Cumulative Probability Function



Contrast discrete UQs with continuous UQs. Continuous UQs have an infinite number of values or so many distinct values that it is difficult to assign probability to each value. Instead, for a continuous UQ we assign probability only to ranges of values.

7.2 CONTINUOUS UNCERTAIN QUANTITIES

Probability Density Functions and Cumulative Probability for Continuous Uncertain Quantities

The total area under a probability density function equals one.

A portion of the area under a density function is a probability.

The height of a density function is not a probability.

The simplest probability density function is the uniform density function.

Case A: Uniform Density

The number of units of a new product that will be sold is an uncertain quantity.

What is the minimum quantity? "1000 units"

What is the maximum quantity? "5000 units"

Are any values in the range between 1000 and 5000 more likely than others? "No"

Represent the uncertainty using a uniform density function.

Technical point: For a continuous UQ, P(X=x) = 0.

For a continuous UQ, probability is non-zero only for a range of values.

For convenience in computation and assessment, we may use a continuous UQ to approximate a discrete UQ, and vice versa.

In the figure below, the range of values is 5000 – 1000 = 4000, which is the width of the total area under the uniform (rectangular) density function. The area of a rectangle is Width * Height = Area, and the area under the uniform density function in the figure must equal 1. So, Height = Area / Base. Here the Base is 5000 – 1000 = 4000 units. Therefore, Height = 1/4000 = 0.00025.

Figure 7.4 Uniform Density Function

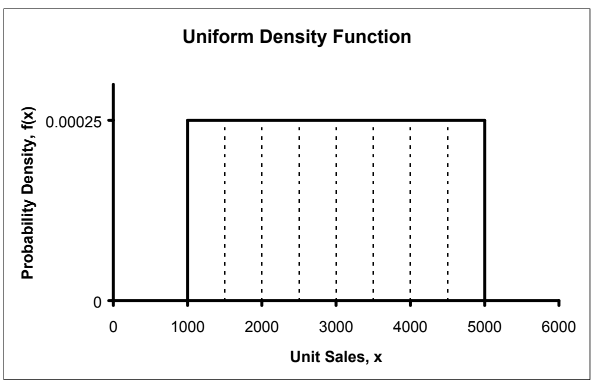
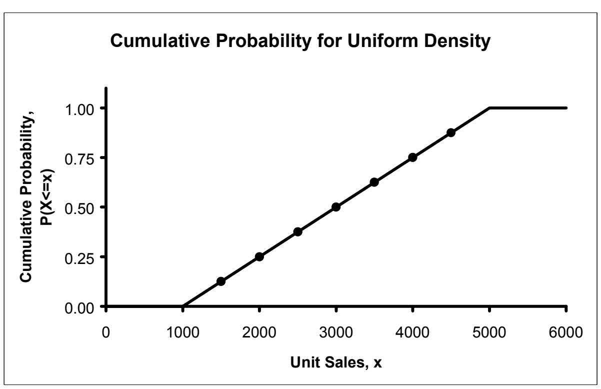


Figure 7.5 Cumulative Probability for Uniform Density



Both probability mass functions (for discrete UQs) and probability density functions (for continuous UQs) have corresponding cumulative probability functions.

It is important to understand the relationship between a density function and its cumulative probability function.

Cumulative probability can be expressed in four ways:

P(X<=x)	probability that UQ X is less than or equal to x	inclusive left -tail
P(X<x)	probability that UQ X is strictly less than x	exclusive left -tail
P(X>=x)	probability that UQ X is greater than or equal to x	inclusive right -tail
P(X>x)	probability that UQ X is strictly greater than x	exclusive right -tail

For continuous UQs the cumulative probability is the same for inclusive and exclusive.

P(X<=x) is the most common type.

What is the probability that sales will be between 3,500 and 4,000 units?

P(3500<=X<=4000) = 0.125

P(3500<=X<=4000) = P(X<=4000) – P(X<=3500) = 0.750 – 0.625 = 0.125

Mathematical observation: The uniform density function is a constant; the corresponding cumulative function (the integral of the constant function) is linear.

Case B: Ramp Density

The number of units of a new product that will be sold is an uncertain quantity.

What is the minimum quantity? "1000 units"

What is the maximum quantity? "5000 units"

Are any values in the range between 1000 and 5000 more likely than others?

"Yes, values close to 5000 are much more likely than values close to 1000."

Represent the uncertainty using a ramp density function.

The area of a triangle is Base * Height / 2, and the area under the ramp density function in the following figure must equal 1. So, Height = 2 / Base. Here, the Base is 5000 – 1000 = 4000 units. Therefore, Height = 2 / 4000 = 0.0005.

Figure 7.6 Ramp Density Function

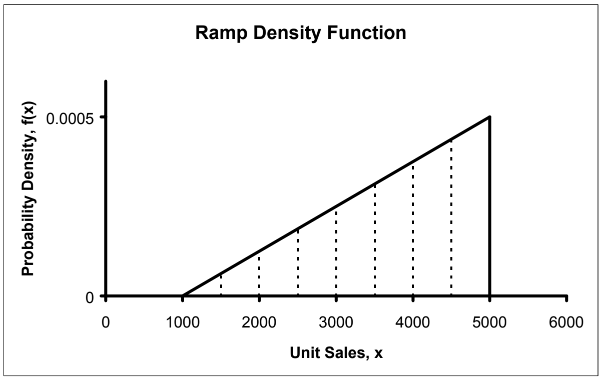
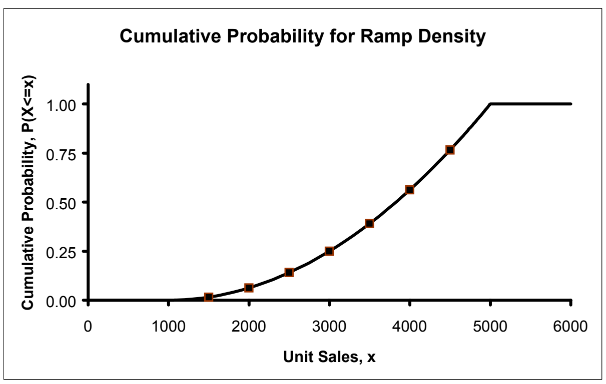


Figure 7.7 Cumulative Probability for Ramp Density



An important observation is that flatter portions of a cumulative probability function correspond to ranges with low probability. Steeper portions of a cumulative probability function correspond to ranges with high probability.

What is the probability that sales will be between 3,500 and 4,000 units?

P(3500<=X<=4000) = 0.171875

P(3500<=X<=4000) = P(X<=4000) – P(X<=3500) = 0.562500 – 0.390625 = 0.171875

The ramp density may not be appropriate for describing uncertainty in many situations, but it is an important building block for the extremely useful triangular density function.

Mathematical observation: The ramp density function is linear; the corresponding cumulative function (the integral of the linear function) is quadratic.

Case C: Triangular Density

The number of units of a new product that will be sold is an uncertain quantity.

What is the minimum quantity? “1000 units”

What is the maximum quantity? “5000 units”

Are any values in the range between 1000 and 5000 more likely than others?

"Yes, values close to 4000 are more likely."

Represent the uncertainty using a triangular density function.

The area of a triangle is Base * Height / 2, and the area under the triangular density function in the following figure must equal 1. So, Height = 2 / Base. Here, the Base is 5000 – 1000 = 4000 units. Thus, Height = 2 / 4000 = 0.0005.

Figure 7.8 Triangular Density Function

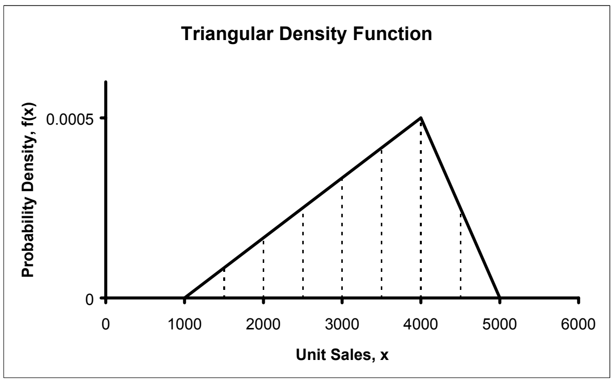
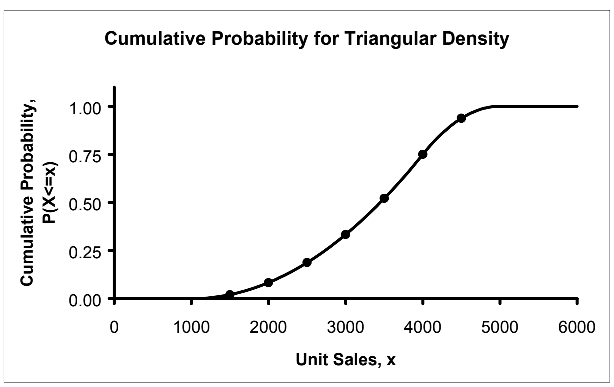


Figure 7.9 Cumulative Probability for Triangular Density



Again, an important observation is that flatter portions of a cumulative probability function correspond to ranges with low probability (the range close to 1000 and the range close to 5000 in the figure). Steeper portions of a cumulative probability function correspond to ranges with high probability (the range close to 4000).

What is the probability that sales will be between 3,500 and 4,000 units?

P(3500<=X<=4000) = 0.229167

P(3500<=X<=4000) = P(X<=4000) – P(X<=3500) = 0.750000 – 0.520833 = 0.229167

The triangular density function is extremely useful for describing uncertainty in many situations. It requires only three inputs: minimum, mode (most likely value), and maximum.

Mathematical observation: The triangular density function has two linear segments, i.e., piecewise linear; the corresponding cumulative function (the integral of each linear function) is two quadratic segments, i.e., piecewise quadratic.